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Boundary-Layer Blockage with Mass Transfer

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Nomenclature

b = mass-transfer parameter, $2(\rho v)_w/\rho_e U_e C_{f_0}$
C_f = skin-friction coefficient
F = streamwise variation of channel geometry, Eq. (12)
f_1 = $-2V\times[1-2/b_{cr_0}]$, Eq. (18)
f_2 = $-2V\times[1+H_1(1-2/b_{cr_0})+(1-M_e^2)(\mathrm{d}lnU_e/\mathrm{d}lnx)]$, Eq. (18)
f_3 = $mS_{cr}\mathrm{d}lnU_e/\mathrm{d}x-\psi\mathrm{d}lnC_{f_0}/\mathrm{d}\theta-2V/C_{f_0}$, Eq. (22)
h = height from wall to channel centerline
H = shape factor, δ^*/θ
M = Mach number
m = constant in Eq. (8), typically 1.75
$\dot m$ = mass flow rate
R_e = Reynolds number
S = pressure gradient parameter, $\theta\mathrm{d}lnU_e/\mathrm{d}x$
$\bar S$ = S/S_{cr}
U = streamwise velocity
v = transverse velocity
V = $(\rho v)_w/2\rho_e U_e\theta$
W = spanwise channel dimension
x = streamwise coordinate
δ^* = displacement thickness
κ = constant in Eq. (2), typically 0.4
ϵ = pressure gradient parameter, Eqs. (6, 7, and 9)
ρ = fluid density
θ = momentum thickness
ψ = generalized skin friction modulator function
ψ_b = $(1-b/b_{cr})^2$, constant pressure, isothermal mass transfer ψ
ψ_p = zero mass transfer, isothermal pressure gradient ψ, Eq. (8)

Subscripts

e = freestream conditions
0 = constant pressure, zero mass transfer
1 = variable pressure, zero mass transfer
cr = critical value
w = wall conditions

CONSIDERATION of boundary-layer blockage effects generally involves data correlations or analytical formulations that utilize the local displacement thickness. Internal flows, for example at the entrance to diffusers, and external flows, on slender bodies whose flowfields are dominated by viscous-induced streamline deflection, follow this treatment. In both of these situations, boundary-layer control (suction or blowing) has application in obtaining efficient designs and/or maintaining structural integrity.

The δ^* distribution may be obtained through simple integral calculations utilizing the momentum equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}+\frac{\theta}{U_e}\frac{\mathrm{d}U_e}{\mathrm{d}x}[H+2-M_e^2]=\frac{C_f}{2}+\frac{(\rho v)_w}{\rho_e U_e} \tag{1}$$

to obtain θ and subsequently $\delta^*=H\theta$.

The results of Ref. 1 provide a method for modeling the effects of pressure gradient and wall temperature on C_f and H in a high-Reynolds-number turbulent flow. The effect of mass transfer on C_f in a constant-pressure isothermal flow also is presented. In addition, an approximate analysis of the influence of mass transfer on C_f in a constant-pressure nonisothermal flow is developed. However, the dependences of H and C_f on mass transfer in a pressure gradient are not addressed.

The purpose of this Note is to point out that, because of the origin of H as a definition based on δ^* and θ, the dependence of H on mass transfer may be exposed directly from considerations of mass continuity. In so doing, it is necessary to utilize only the physical interpretation of δ^* and the fact that mass transfer appears as an explicit term in the momentum equation. A prerequisite to this approach is the development of an appropriate C_f model with mass transfer in a pressure gradient.

Large mass transfer rates are excluded from consideration in order to obtain a closed-form "linearized" solution. This is of particular relevance in diffuser blockage determination, wherein a small amount of system power is diverted to apply suction along a short segment of upstream boundary layer such that the increased diffuser recovery results in a net increase in system efficiency.

The zero-mass-transfer, constant-pressure shape factor is obtained[1] from the relationship

$$H_0=1+\sqrt{2C_{f_0}}/(\kappa-\sqrt{2C_{f_0}}) \tag{2}$$

which extends to variable pressure flows through use of the following endpoint fitting formula:

$$(H_1-1)/(H_0-1)=[(H_{cr}-1)/(H_0-1)]^{\bar S} \tag{3}$$

Both H_0 and H_{cr} are affected by the degree of nonisothermality, as indicated by the results of Ref. 1.

A variety of skin-friction models for constant-pressure flows are available. The formula employed in Ref. 1,

$$C_{f_0}=2.0/(2.5lnRe_\theta+3.8)^2 \tag{4}$$

provides generally useful results for smooth surfaces. Roughness amplification of this value may be obtained from the development in Ref. 2. Inclusion of pressure gradient effects is possible utilizing results from previously computed flows (see Fig. 28 of Ref. 1).

For constant-pressure flows, the combined effects of mass transfer and nonisothermality on C_f are approximated[1] by a superposition of their separate influences. The assumption is made here that the same can be done in variable-pressure flows, and, therefore, isothermal flow is considered for notational clarity.

In constant-pressure flows, the effect of mass transfer on C_f may be written as

$$C_f=C_{f_0}\times\psi_b \tag{5}$$

where ψ_b is obtained from the integration of Eq. (4.2) of Ref. 1. In addition, $b_{cr}=4.0$, as indicated from the integration when $\psi=0$.

In the presence of pressure gradients, Eq. (5) is inappropriate, and inclusion of a *weak* pressure gradient

Received May 18, 1977; revision received July 12, 1977.
Index category: Boundary Layers and Convective Heat Transfer – Turbulent.

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dependence (ϵ) may be effected by collocation of the integral [Eq. (4.2) of Ref. 1] and results in

$$\psi+\epsilon=(1-b/b_{cr_0})^2 \tag{6}$$

and

$$b_{cr}=4(1\pm\sqrt{|\epsilon|}) \tag{7}$$

where the plus or minus sign applies to favorable and adverse pressure gradients, respectively.

In order to ascertain an appropriate functional dependence for ϵ, the results of Fig. 28 of Ref. 1 have been curve-fit to give

$$\psi_p=(1-\tilde{S})\ \exp[\log_{10}Re_\theta^{1/m}] \tag{8}$$

Therefore, for small pressure gradients, Eqs. (6) and (8) provide the relationship

$$\epsilon=\tilde{S}\ \log_{10}Re_\theta^{1/m} \tag{9}$$

inasmuch as, when $b=0$, $\psi=\psi_p=1-\epsilon$. With the basic dependence of H on S and C_f on S and b available, it is possible to develop the dependence of H on b through manipulation of the continuity equation.

The mass flux through the boundary layer is integrated from the wall to the centerline of a two-dimensional channel bounded by the porous surfaces of interest. The resulting equation is

$$\dot{m}=\rho_e U_e W(h-H\theta) \tag{10}$$

Total mass flow through the channel is controlled by the wall mass transfer according to

$$\frac{d\dot{m}}{dx}=(\rho v)_w W \tag{11}$$

Combining Eqs. (10) and (11), the following relationship ensues:

$$\theta\frac{dH}{dx}+H\frac{d\theta}{dx}=-\frac{(\rho v)_w}{\rho_e U_e}$$

$$-\frac{\dot{m}|_{b=0}+\int(\rho v)_w W dx}{\rho_e U_e W}\times\frac{d\ln(\rho_e U_e W)}{dx}+F \tag{12}$$

Consider the application of Eq. (12) to two hypothetical configurations with nearly identical geometry, entrance mass flow rates, and freestream property distributions, one with mass transfer and one without. Upon applying Eq. (12) to each channel and subtracting, the result is

$$\frac{1}{H}\left(1-\frac{\Delta\theta}{\theta}\right)\frac{d\Delta H}{dx}+\frac{1}{\theta}\left(1-\frac{\Delta H}{H}\right)\Delta\frac{d\theta}{dx}+\frac{\Delta\theta}{\theta}\frac{d\ln H}{dx}$$

$$+\frac{\Delta H}{H}\frac{d\ln\theta}{dx}=\frac{(\rho v)_w H^{-1}}{\rho_e U_e\theta}\left(1+(1-M_e^2)\frac{d\ln U_e}{d\ln x}\right)+\Delta F \tag{13}$$

where it has been assumed that the channel width and mass transfer are approximately constant, and the freestream flowfield is isentropic. In order for the channel geometries and flowfields to be nearly identical, the mass transfer must be small (in conformance with the determination of mass transfer in a pressure gradient), such that $\Delta\theta/\theta$, $\Delta H/H$, and ΔF are negligible. Enforcing these restrictions, Eq. (13) becomes

$$\frac{1}{H}\frac{d\Delta H}{dx}+\frac{1}{\theta}\Delta\frac{d\theta}{dx}=-2VH^{-1}\left[1+(1-M_e^2)\frac{d\ln U_e}{d\ln x}\right] \tag{14}$$

In order to evaluate the second term in Eq. (14), Eq. (1) also may be applied to the two flowfields of interest in the following manner:

$$\Delta\frac{d\theta}{dx}+\frac{H\theta}{U_e}\frac{dU_e}{dx}\left\{\frac{\Delta\theta}{\theta}\left[1+(2-M_e^2)H^{-1}\right]+\frac{\Delta H}{H}\left[1-\frac{\Delta\theta}{\theta}\right]\right\}$$

$$=\frac{\psi}{2}\left\{\frac{\Delta\theta}{\theta}\left[1+\frac{\Delta\psi}{\psi}\right]\frac{dC_{f_0}}{d\ln\theta}+C_{f_0}\frac{\Delta\psi}{\psi}\right\}+\frac{(\rho v)_w}{\rho_e U_e} \tag{15}$$

Utilizing the linearization assumption and combining Eq. (15) with Eq. (16), the resultant expression becomes

$$\Delta\frac{d\theta}{dx}=\frac{(\rho v)_w}{\rho_e U_e}\left[1-\frac{2}{b_{cr_0}}\right] \tag{16}$$

Combining Eqs. (14) and (16) results in

$$\frac{d\Delta H}{dx}+2HV\left[1-\frac{2}{b_{cr_0}}\right]=-2V\left[1+(1-M_e^2)\frac{d\ln U_e}{d\ln x}\right] \tag{17}$$

which integrates to give

$$H=H_1+\ \exp(-\int f_1 dx)\int f_2\ \exp(\int f_1 dx)dx \tag{18}$$

For constant-pressure flows, assuming $b_{cr_0}=4$, Eq. (18) may be written as

$$H=H_0+\exp(-\int V dx)\int V(H_0+2)\ \exp(-\int V dx)dx \tag{19}$$

Throughout this exercise, it has been assumed that the pressure gradient is weak in order to obtain Eqs. (6, 9, 14, and 16). Therefore, inclusion of nonnegligible $\Delta\theta/\theta$ and $\Delta H/H$ effects in the third and fourth terms of Eq. (13) may be approached to extend the range of validity of Eq. (19). Assuming that ΔF is still small while supporting the same external flowfield, Eq. (15) for small pressure gradients and finite $\Delta\theta/\theta$ and $\Delta H/H$ becomes

$$\Delta\frac{d\theta}{dx}=\frac{\psi}{2}\Delta\theta\frac{dC_{f_0}}{d\theta}+\frac{C_{f_0}}{2}\Delta\psi+\frac{(\rho v)_w}{\rho_e U_e} \tag{20}$$

Utilizing Eqs. (6) and (9), $\Delta\psi$ may be written as

$$\Delta\psi=\frac{-\tilde{S}}{m}\frac{\Delta\theta}{\theta}-\frac{2b}{b_{cr_0}} \tag{21}$$

Therefore, Eq. (20) becomes

$$\frac{d\Delta\theta}{dx}+f_3\frac{C_{f_0}}{2}\Delta\theta=V\theta_0 \tag{22}$$

which may be utilized to provide expressions for $\Delta d\theta/dx$ and $\Delta\theta$ in Eq. (13). The expressions for $d\ln\theta/dx$ and $d\ln H/dx$ then are evaluated at the zero-mass-transfer condition, an approximation consistent with assuming that $\Delta\theta/\theta$ and $\Delta H/H$ are nonzero and yet still small compared to unity.

Acknowledgment

This work was carried out as part of the research sponsored by the U.S. Energy Research and Development Administration, Fossil Energy Division, MHD Division, under Contract EX-76-C-01-2243.

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